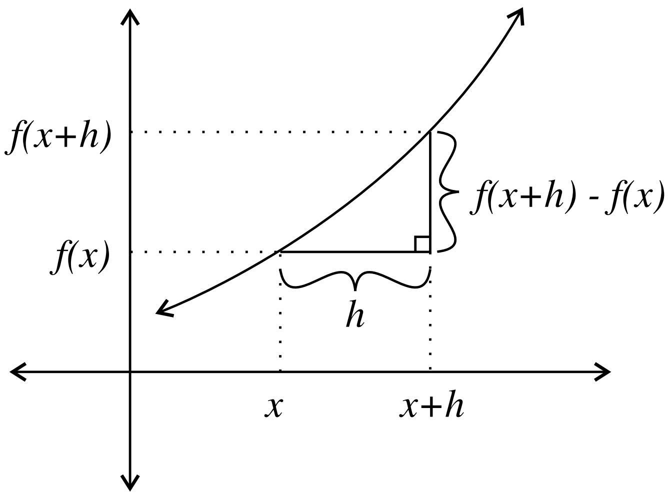
Worksheet 3.8 Introduction to Differentiation

Section 1 Definition of Differentiation

Differentiation is a process of looking at the way a function changes from one point to another. Given any function we may need to find out what it looks like when graphed. Differentiation tells us about the slope (or rise over run, or gradient, depending on the tendencies of your favourite teacher). As an introduction to differentiation we will first look at how the derivative of a function is found and see the connection between the derivative and the slope of the function.



Given the function $f(x)$, we are interested in finding an approximation of the slope of the function at a particular value of x. If we take two points on the graph of the function which are very close to each other and calculate the slope of the line joining them we will be approximating the slope of $f(x)$ between the two points. Our x-values are x and $x+h$, where h is some small number. The y-values corresponding to x and $x+h$ are $f(x)$ and $f(x+h)$. The slope m of the line between the two points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are the two points. In our case, we have the two points $(x, f(x))$ and $(x+h, f(x+h))$. So the slope of the line joining them is given by

$$\begin{aligned} m &= \frac{f(x+h) - f(x)}{x+h-x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

Example 1 : Let $f(x) = x^3$. Find the slope of the line joining $(x, f(x))$ and $(x+h, f(x+h))$. From our definitions,

$$\begin{aligned} m &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{(x+h)^3 - x^3}{h} \\ &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= 3x^2 + 3xh + h^2 \end{aligned}$$

Example 2 : Let $f(x) = 2x + 5$. Find the slope of the line joining the points $(1, f(1))$ and $(1.01, f(1.01))$.

$$\begin{aligned} m &= \frac{f(1.01) - f(1)}{1.01 - 1} \\ &= \frac{7.02 - 7}{0.01} \\ &= \frac{0.02}{0.01} \\ &= 2 \end{aligned}$$

as expected since the gradient of $y = 2x + 5$ is 2.

Example 3 : Let $f(x) = x^2$. Find the slope of the line joining $(x, f(x))$ and $(x+h, f(x+h))$ if $h = 0.1$ and $x = 1$.

$$\begin{aligned} m &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{f(1+0.1) - f(1)}{0.1} \\ &= \frac{f(1.1) - f(1)}{0.1} \\ &= \frac{(1.1)^2 - (1)^2}{0.1} \\ &= \frac{0.21}{0.1} \\ &= 2.1 \end{aligned}$$

The smaller that h gets to zero, the closer x and $x+h$ get to each other, and consequently the better m approximates the slope of the function at the point $(x, f(x))$. So we look at what happens when we take the limit as $h \to 0$ in the slope formula and we call this the derivative $f'(x)$. So

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Notice that $f'(x)$ is the derivative only if the limit exists. If the limit does not exist at particular x-values then we say that the function is not differentiable at those x-values.

Example 4 : Find the derivative of $f(x) = x^2 + 3$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 + 3 - (x^2 + 3)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3 - x^2 - 3}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

Note: There are other notations for the derivative of a function in x. The most common are $f'(x)$ and $\frac{df}{dx}$. If $y = f(x)$, we also use $y' = f'(x)$ or $\frac{dy}{dx}$ to refer to the derivative.

Example 5 : Find the derivative of the function $f(x) = 2x + 5$ at $x = 1$.

$$\begin{aligned} f(x+h) &= 2(x+h) + 5 \\ f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{2(1+h) + 5 - 7}{h} \\ &= \lim_{h \to 0} \frac{2h}{h} \\ &= \lim_{h \to 0} 2 \\ &= 2 \end{aligned}$$

Example 6 : Find the derivative of $y = |x|$ at $x = 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{|0+h| - |0|}{h} \\ &= \lim_{h\to 0} \frac{|h|}{h} \end{aligned}$$

Recall that

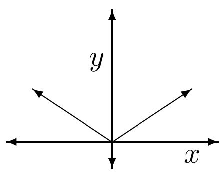
$$|h| = \begin{cases} h & \text{when } h \geq 0 \\ -h & \text{when } h < 0 \end{cases}$$

The absolute-value sign prevents us from simply canceling. Let's look at what $\frac{|h|}{h}$ equals. h could be very small and negative, in which case $f'(0) = -1$. Or it could be very small and positive, in which case $f'(0) = 1$. That is

if $h < 0$, then $f'(0) = -1$
if $h > 0$, then $f'(0) = 1$

So the limit does not exist as $h \to 0$ since we get a different value for the limit depending upon whether or not we are close to zero on the negative side or the positive side. Therefore the derivative of $f(x) = |x|$ does not exist at $x = 0$.

Look at the graph of $y = |\, x|$.



The pointed part at $x = 0$ shows a rapid and abrupt change of slope. Functions that have sharp points on their graphs do not have derivatives at these points, although they may have a derivative everywhere else. The function $f(x) = |x|$ is not differentiable at $x = 0$, although it is continuous there.

Exercises:

1. Using the method outlined above, find $f'(x)$ for each of the following functions. That is, use
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
   (a) $f(x) = x^2 + 2$
   (b) $f(x) = 3x - 5$
   (c) $f(x) = 3 - x^2$
   (d) $f(x) = 4x + 5$
   (e) $f(x) = 2 - x$